Ruelle's inequality and Pesin's entropy formula for the geodesic flow on negatively curved noncompact manifolds

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RUELLE'S INEQUALITY AND PESIN'S ENTROPY FORMULA FOR THE GEODESIC FLOW ON NEGATIVELY CURVED NONCOMPACT MANIFOLDS

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ABSTRACT. In this paper we study different notions of entropy of measure-preserving dynamical systems defined on noncompact spaces. We see that some classical results for compact spaces remain partially valid in this setting. We define a new kind of entropy for dynamical systems defined on noncompact Riemannian manifolds, which satisfies similar properties to the classical ones. As an application, we prove Ruelle's inequality and Pesin's entropy formula for the geodesic flow in manifolds with pinched negative sectional curvature.

1. INTRODUCTION

1.1. **Motivation and statements of main results.** *Ruelle's inequality* [Rue78] is an important result in ergodic theory for smooth dynamical systems relating two fundamental concepts: *measure-theoretic entropy* and *Lyapunov exponents*. It precisely states that if $f : M \to M$ is a C^1-diffeomorphism of a compact Riemannian manifold and μ is an f-invariant probability measure on M, then the measure-theoretic entropy $h_\mu(f)$ is bounded from above by the sum of the positive Lyapunov exponents, i.e.

$$h_\mu(f) \leq \int \sum_{\lambda_j(x)>0} \lambda_j(x) \dim(E_j(x)) d\mu(x), \tag{1.1}$$

where $\{\lambda_j(x)\}$ is the set of Lyapunov exponents at $x \in M$ and $\dim(E_j(x))$ is the multiplicity of $\lambda_j(x)$.

Once inequality (1.1) is established, the question about the equality case, known as *Pesin's entropy formula*, arises naturally. For $C^{1+\alpha}$-diffeomorphisms F. Ledrappier and L.-S. Young showed in [LY85] that an f-invariant probability measure verifies Pesin's entropy formula if and only if it is absolutely continuous along unstable manifolds (see also [Pes77], [LS82] and [Led84a]).

Surprisingly, Ruelle's inequality can fail to be true on noncompact manifolds. To be more precise, in [Riq15] we proved that there exist smooth dynamical systems having (arbitrary) positive measure-theoretic entropy whereas the sum of the positive Lyapunov exponents is equal to zero. This implies in particular that, even for smooth enough dynamical systems, Ruelle's inequality is not always satisfied when the manifold is not compact. Therefore, it becomes an important question to investigate in which (noncompact) manifolds this inequality is verified.

Date: June 15, 2016.

2010 *Mathematics Subject Classification.* Primary 37A05, 37A35, 37D40; Secondary 37D10, 37D35.

Key words and phrases. Entropy, Geodesic flow, Lyapunov exponents, Gibbs measure.

The aim of this paper is to prove Ruelle's inequality and Pesin's entropy formula for the geodesic flow on the unit tangent bundle of a negatively curved noncompact manifold satisfying reasonable assumptions. The necessity of these assumptions will be clarified in the core of the text.

Theorem 1.1. *Let X be a complete Riemannian manifold with dimension at least 2 and pinched negative sectional curvature. Assume that the derivatives of the sectional curvature are uniformly bounded. Then, for every (g^t)-invariant probability measure μ on T^1X, we have*

$$h_\mu((g^t)) \leq \int \sum_{\lambda_j(v)>0} \lambda_j(v) \dim(E_j(v)) d\mu(v).$$

Our second main result treats the equality case on Ruelle's inequality as in [LY85]. No additional assumptions to those of Theorem 1.1 are needed.

Theorem 1.2. *Let X be a complete Riemannian manifold with dimension at least 2 and pinched negative sectional curvature. Assume that the derivatives of the sectional curvature are uniformly bounded. Let μ be a (g^t)-invariant probability measure on T^1X. Then μ has absolutely continuous conditional measures on unstable manifolds if and only if*

$$h_\mu((g^t)) = \int \sum_{\lambda_j(v)>0} \lambda_j(v) \dim(E_j(v)) d\mu(v).$$

1.2. **Structure of the paper.** Section 2 will be devoted to study deeply the concept of entropy. We are particularly interested in the interaction between two notions, namely measure-theoretic entropy and Brin-Katok local entropy for continuous transformations defined on complete metric spaces. When the space is compact, these two values coincide [BK83]. We extend this equality to the noncompact case as follows. Consider a continuous transformation $T : M \to M$ of a complete Riemannian manifold. Let d be the Riemannian distance on M. Recall that for every $n \geq 1$ and $r > 0$, the (n, r)-dynamical ball centered at $x \in M$, denoted by $B_n(x, r)$, is the set of points $y \in M$ satisfying $d(T^i x, T^i x) < r$ for all $0 \leq i \leq n - 1$.

Theorem 1.3. *Let $T : M \to M$ be a Lipschitz transformation of a complete Riemannian manifold and μ an ergodic T-invariant probability measure on M. Then*

$$h_\mu(T) = \sup_K \operatorname*{ess\,sup}_{x\in K} \lim_{r\to 0} \liminf_{n\to\infty} -\frac{1}{n} \log \mu(B_n(x, r)), \tag{1.2}$$

where the supremum is taken over all the compact subsets K of M having positive μ-measure.

In Section 3 we observe the following phenomenon. The exponential decay of the volume of generic dynamical balls for C^1-diffeomorphisms at the level of the tangent space of the manifold is equal to the sum of the positive Lyapunov exponents (see Theorem 3.7). Inspired by this result, we relate the measure-theoretic entropy with the exponential decay of the volume of classical dynamical balls as in equation (1.2) but taking the Riemannian measure $\mathcal{L}$ instead of μ. More precisely, we prove

Theorem 1.4. *Let $T : M \to M$ be a continuous transformation of a complete Riemannian manifold, preserving an ergodic T-invariant probability measure μ. Then*

$$h_\mu(T) \leq \sup_K \operatorname*{ess\,sup}_{x\in K} \lim_{r\to 0} \limsup_{\substack{n\to\infty \\ T^n x\in K}} -\frac{1}{n} \log \mathcal{L}(B_n(x,r)), \tag{1.3}$$

where the supremum is taken over all the compact subsets K of M having positive μ-measure.

It is interesting to remark that the measure μ appears in the right-hand term of inequality (1.3) only when we consider the μ-essential supremum of the exponential decays of the volumes of dynamical balls. Therefore, this result allows to relate a purely dynamical value, measure-theoretic entropy, with a dynamical/geometrical one, exponential decay of the Riemannian measure of a dynamical ball. As a consequence, it is natural to ask ourselves if Ruelle's inequality is true under "nice" linearization assumptions. Despite not having a conclusive result in this direction for arbitrary diffeomorphisms, under the assumptions of Theorem 1.1 we can estimate the volume of a dynamical ball for the geodesic flow on the unit tangent bundle of a Riemannian manifold X with negative curvature (see Section 4). More precisely, the Liouville measure, which is the Lebesgue measure on $M = T^1X$, satisfies the Gibbs property for the *geometric potential* (see Proposition 4.2). Using this last fact together a well-known relation between the geometric potential and the positive Lyapunov exponents, Ruelle's inequality is proved.

Finally, using the strategies exhibited in [LS82], [Led84a] to prove Pesin's entropy formula for $C^{1+\alpha}$-diffeomorphisms in the compact case, adapted as in [OP04] to the case of the geodesic flow in the noncompact setting, we prove Theorem 1.2. Some direct corollaries of Theorem 1.1 are stated at the end of the paper.

Acknowledgements. *I am very grateful to my adviser Barbara Schapira for many helpful discussions from the very beginning of this work. I would also like to thank Samuel Tapie for his inspiring remarks on some results of this paper, and Godofredo Iommi for some nice suggestions that improved the presentation of this work.*

2. Preliminaries

In this section we consider a probability space $(X, \mathcal{B}, \mu)$ and a measurable transformation $T : X \to X$ preserving the measure μ. Recall that μ is *ergodic* if every T-invariant measurable set $A \subset X$ satisfies $\mu(A) \in \{0, 1\}$.

2.1. **Measure-theoretic entropy.**

Let $\mathcal{P}$ be a countable measurable partition of X. The entropy of $\mathcal{P}$ with respect to μ, denoted by $H_\mu(\mathcal{P})$, is defined as

$$H_\mu(\mathcal{P}) = -\sum_{P\in\mathcal{P}} \mu(P) \log \mu(P).$$

For all $n \geq 0$, define the partition $\mathcal{P}^n$ as the measurable partition consisting of all possible intersections of elements of $T^{-i}\mathcal{P}$, for all $i = 0, ..., n-1$. The entropy of T with respect to the partition $\mathcal{P}$ is then defined as the limit

$$h_\mu(T, \mathcal{P}) = \lim_{n\to\infty} \frac{1}{n} H_\mu(\mathcal{P}^n).$$

The *measure-theoretic entropy of* T, *with respect to* μ, is the supremum of the entropies $h_\mu(T,\mathcal{P})$ over all measurable finite partitions $\mathcal{P}$ of X, i.e.

$$h_\mu(T) = \sup_{\mathcal{P} \text{ finite}} h_\mu(T,\mathcal{P}).$$

2.2. **Katok δ-entropies.** Suppose now that (X,d) is a metric space, $T : X \to X$ is a continuous transformation and μ is a Borel T-invariant probability measure on X. For every $n \geq 1$ the dynamical distance d_n is defined by

$$d_n(x,y) = \max_{0\leq i\leq n-1} d(T^i x, T^i y), \quad \text{for all} \quad x,y \in X.$$

The (n,r)-dynamical ball centered at x, denoted by $B_n(x,r)$, is the r-ball centered at x for the dynamical distance d_n. Note that since T is continuous, the (n,r)-dynamical balls are open subsets of X. Let A be a subset of X. A (n,r)-*covering* of A is a covering of A by (n,r)-dynamical balls. A (n,r)-*separated* set in A is a subset E of A such that for every $x,y \in E$, if $x \neq y$ then $d_n(x,y) \geq r$.

Definition 2.1. Let $K \subset X$ be a compact set. Denote by

(1) $N(n,r,K)$ the minimal cardinality of a (n,r)-covering of K, i.e. a covering of K by (n,r)-balls, and
(2) $S(n,r,K)$ the maximal cardinality of a (n,r)-separated set in K.

Lemma 2.2 below is classical (see for instance [Wal82, page 169]). It says that the cardinalities $N(n,r,K)$ and $S(n,r,K)$ are comparable.

Lemma 2.2. *Let $n \geq 1$ and $r > 0$. Then for all compact subsets $K \subset X$, we have*

$$N(n,r,K) \leq S(n,r,K) \leq N(n,r/2,K).$$

Recall that Bowen's definition of topological entropy of a continuous transformation on a compact metric space is the following:

$$\begin{aligned} h_{top}(T) &= \lim_{r\to 0} \limsup_{n\to\infty} \frac{1}{n} \log N(n,r,X) \\ &= \lim_{r\to 0} \limsup_{n\to\infty} \frac{1}{n} \log S(n,r,X). \end{aligned}$$

From the measure point of view, Katok proposed the following definition of entropy in [Kat80]. For every $0 < \delta < 1$, denote by $N_\mu(n,r,\delta)$ the minimal cardinality of a (n,r)-covering of a set of μ-measure greater than $1-\delta$. Observe that this number is finite since every compact subset of measure greater than $1-\delta$ admits a finite (n,r)-covering.

Definition 2.3. Let $0 < \delta < 1$. The *lower* and *upper δ-entropies* relative to μ, denoted respectively by $\underline{h}^\delta_\mu(T)$ and $\overline{h}^\delta_\mu(T)$, are defined as

$$\underline{h}^\delta_\mu(T) = \lim_{r\to 0} \liminf_{n\to\infty} \frac{1}{n} \log N_\mu(n,r,\delta)$$

and

$$\overline{h}^\delta_\mu(T) = \lim_{r\to 0} \limsup_{n\to\infty} \frac{1}{n} \log N_\mu(n,r,\delta).$$

Proposition 2.4. *Let $0 < \delta_2 \leq \delta_1 < 1$, then*

$$\underline{h}^{\delta_1}_\mu(T) \leq \underline{h}^{\delta_2}_\mu(T) \quad \textit{and} \quad \overline{h}^{\delta_1}_\mu(T) \leq \overline{h}^{\delta_2}_\mu(T).$$

Proof. We define $\mathcal{B}_i$, for $i = 1, 2$, by $\mathcal{B}_i = \{B : \mu(B) > 1 - \delta_i\}$. Since $\mathcal{B}_2 \subset \mathcal{B}_1$, we obtain

$$\begin{aligned} \underline{h}_{\mu}^{\delta_1}(T) &= \lim_{r\to 0} \liminf_{n\to\infty} \frac{1}{n} \log \min\{N(n, r, B) : B \in \mathcal{B}_1\} \\ &\leq \lim_{r\to 0} \liminf_{n\to\infty} \frac{1}{n} \log \min\{N(n, r, B) : B \in \mathcal{B}_2\} \\ &= \underline{h}_{\mu}^{\delta_2}(T) \end{aligned}$$

The other inequality can be proved similarly. □

Suppose that X is a compact metric space. Katok proved in [Kat80, Theorem 1.1] that the lower and upper δ-entropies are equal and coincide with the measure-theoretic entropy. In his proof the assumption of compacity for X is only used to show that $\overline{h}_{\mu}^{\delta}(T) \leq h_{\mu}(T)$. The other inequality is only based on the fact that $h_{\mu}(T)$ can be approximate by entropies $h_{\mu}(T, \mathcal{P})$, with respect to a partition $\mathcal{P}$ satisfying $\mu(\partial\mathcal{P}) = 0$, where $\partial\mathcal{P}$ is the union of the boundaries of the elements of $\mathcal{P}$[1]. So one can conclude the following:

Theorem 2.5 (Katok)**.** *Let X be a complete metric space and let $T : X \to X$ be a continuous transformation. If μ is an ergodic T-invariant probability measure, then for all $0 < \delta < 1$, we have*

$$h_{\mu}(T) \leq \underline{h}_{\mu}^{\delta}(T).$$

2.3. **Local entropies of Brin-Katok.** The aim of this subsection is to understand some relations between previous notions of entropy and local entropy. The notion of local entropy was introduced by Brin and Katok in [BK83].

Definition 2.6. The *lower* and *upper local entropies* of T relative to μ, denoted respectively by $\underline{h}_{\mu}^{loc}(T)$ and $\overline{h}_{\mu}^{loc}(T)$, are defined as

$$\underline{h}_{\mu}^{loc}(T) = \operatorname*{ess\,inf}_{x\in X} \lim_{r\to 0} \liminf_{n\to\infty} -\frac{1}{n} \log \mu(B_n(x, r))$$

and

$$\overline{h}_{\mu}^{loc}(T) = \operatorname*{ess\,sup}_{x\in X} \lim_{r\to 0} \limsup_{n\to\infty} -\frac{1}{n} \log \mu(B_n(x, r)).$$

Lemma 2.7. *Let X be a complete metric space and let $T : X \to X$ be a continuous transformation. If μ is an ergodic T-invariant probability measure on X, then*

$$\underline{h}_{\mu}^{loc}(T) = \int \lim_{r\to 0} \liminf_{n\to\infty} -\frac{1}{n} \log \mu(B_n(x, r)) d\mu(x)$$

and

$$\overline{h}_{\mu}^{loc}(T) = \int \lim_{r\to 0} \limsup_{n\to\infty} -\frac{1}{n} \log \mu(B_n(x, r)) d\mu(x).$$

Proof. For the sake of simplicity, for each $x \in X$ we denote by $\underline{h}_{\mu}^{loc}(T, x)$ the limit

$$\lim_{r\to 0} \liminf_{n\to\infty} -\frac{1}{n} \log \mu(B_n(x, r)).$$

[1]The validity of this inequality in the noncompact case has been also remarked in [GK01] to compute the topological entropy of the geodesic flow in the modular surface.

Since $T(B_{n+1}(x,r)) \subset B_n(Tx,r)$ for every $n \geq 1$, the T-invariance of μ implies that

$$\underline{h}_{\mu}^{loc}(T,Tx) \leq \underline{h}_{\mu}^{loc}(T,x), \tag{2.1}$$

for every $x \in X$. Define $\underline{\eta}(x)$ as $\underline{\eta}(x) = \inf_{k\geq 0} \underline{h}_{\mu}^{loc}(T,T^k x)$. By definition $\underline{\eta}$ is a T-invariant function, so it is μ-a.e. constant equal to some constant $\underline{\eta}(\mu)$. Note that $\underline{\eta}(\mu)$ is also equal to the essential infimum of $\underline{h}_{\mu}^{loc}(T,x)$. On the other hand, inequality (2.1) implies that

$$\underline{\eta}(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \underline{h}_{\mu}^{loc}(T,T^k x),$$

for every $x \in X$. Using Birkhoff's Ergodic Theorem, we conclude that

$$\underline{\eta}(\mu) = \int \underline{h}_{\mu}^{loc}(T,x) d\mu(x),$$

which is exactly the first desired equality. The second equality follows from the same strategy by considering the supremum instead of the infimum in every involved term. □

As said above, when X is a compact metric space, the lower and upper local entropies coincide and are equal to the measure-theoretic entropy. As in the case of δ-entropies, only one inequality in the proof requires the compactness assumption on X. Thus, in the general case following exactly the proof in [BK83], we obtain

Theorem 2.8 (Brin-Katok). *Let X be a complete metric space and let $T : X \to X$ be a continuous transformation. If μ is an ergodic T-invariant probability measure on X, then*

$$h_{\mu}(T) \leq \underline{h}_{\mu}^{loc}(T). \tag{2.2}$$

Following Ledrappier [Led13], we now prove the equality case in (2.2) for Lipschitz transformations defined on (noncompact) Riemannian manifolds.

Theorem 2.9. *Let $T : M \to M$ be a Lipschitz transformation of a complete Riemannian manifold and μ an ergodic T-invariant probability measure. Then*

$$h_{\mu}(T) = \underline{h}_{\mu}^{loc}(T).$$

Proof. As consequence of Theorem 2.8, we only need to prove that $h_{\mu}(T) \geq \underline{h}_{\mu}^{loc}(T)$. This follows from Proposition 2.10 below.

Proposition 2.10 (Ledrappier [Led13], Proposition 6.3). *Let $T : M \to M$ be a Lipschitz transformation of a complete Riemannian manifold and μ an ergodic T-invariant probability measure. Then, for every compact set $K \subset M$ such that $\mu(K) > 0$ and all $0 < r < 1$, there exists a partition $\widehat{\mathcal{P}}$ of K with finite entropy such that, if $\mathcal{P} = \widehat{\mathcal{P}} \cup \{M \setminus K\}$, then for μ-almost every $x \in K$ the sequence $(n_k)_{k\geq 0}$ of return times of x into K satisfies*

$$\mathcal{P}^{n_k}(x) \subset B_{n_k}(x,r),$$

for all $k \geq 0$.

Let $\mathcal{P}$ be as in Proposition 2.10. Using the ergodicity of μ, for μ-a.e. $x \in M$ there exists an integer $k < 0$ such that $T^k x \in K$. We know from the construction of $\mathcal{P}$ that the inclusion $\mathcal{P}^n(T^k x) \subset B_n(T^k x, r)$ is satisfied for infinitely many integers n. In particular, we deduce that $\mathcal{P}^{n+k}(x) \subset T^{-k} B_n(T^k x, r)$ is also satisfied for infinitely many n's. Therefore

$$\begin{aligned}\lim_{n\to\infty} -\frac{1}{n}\log\mu(\mathcal{P}^n(x)) &\geq \liminf_{n\to\infty} -\frac{1}{n}\log\mu(T^{-k}B_n(T^k x, r)) \\ &= \liminf_{n\to\infty} -\frac{1}{n}\log\mu(B_n(T^k x, r)) \\ &\geq \liminf_{n\to\infty} -\frac{1}{n}\log\mu(B_n(x, r)).\end{aligned}$$

Thus, for μ-a.e. $x \in M$, we have

$$\limsup_{n\to\infty} -\frac{1}{n}\log\mu(\mathcal{P}^n(x)) \geq \liminf_{n\to\infty} -\frac{1}{n}\log\mu(B_n(x, r)). \tag{2.3}$$

Recall that $\mathcal{P}$ has finite entropy, then the left-hand side of inequality (2.3) is μ-almost everywhere a limit because of Shannon-McMillan-Breiman Theorem. By Lemma 2.7 and Monotone Convergence Theorem, for every $\varepsilon > 0$ there exists $r_0 > 0$ such that for all $0 < r < r_0$, we have

$$\int \liminf_{n\to\infty} -\frac{1}{n}\log\mu(B_n(x, r))d\mu(x) \geq \underline{h}_\mu^{loc}(T) - \varepsilon.$$

Using inequality (2.3) together with Shannon-McMillan-Breiman Theorem, we obtain

$$\begin{aligned}h_\mu(T) &\geq h_\mu(T, \mathcal{P}) = \int \lim_{n\to\infty} -\frac{1}{n}\log\mu(\mathcal{P}^n(x))d\mu(x) \\ &\geq \int \liminf_{n\to\infty} -\frac{1}{n}\log\mu(B_n(x, r))d\mu(x) \geq \underline{h}_\mu^{loc}(T) - \varepsilon.\end{aligned}$$

Hence, the desired inequality follows when $\varepsilon \to 0$. □

Lemma 2.11. *Let $T : M \to M$ be a Lipschitz transformation of a complete Riemannian manifold and μ an ergodic T-invariant probability measure. Then*

$$\sup_K \operatorname*{ess\,sup}_{x\in K} \lim_{r\to 0} \limsup_{\substack{n\to\infty \\ T^n x \in K}} -\frac{1}{n}\log\mu(B_n(x, r)) \leq h_\mu(T).$$

Proof. Since T is Lipschitz, Proposition 2.10 and Shannon-McMillan-Breiman Theorem imply that, for every compact set $K \subset M$ such that $\mu(K) > 0$, and every $0 < r < 1$, we have

$$\limsup_{\substack{n\to\infty \\ T^n x \in K}} -\frac{1}{n}\log\mu(B_n(x, r)) \leq h_\mu(T, \mathcal{P}) \leq h_\mu(T),$$

for μ-a.e. $x \in K$. This implies the desired inequality. □

We now prove Theorem 1.3. Observe first that we always have the following inequality

$$\underline{h}_\mu^{loc}(T) \leq \sup_K \operatorname*{ess\,sup}_{x\in K} \lim_{r\to 0} \limsup_{\substack{n\to\infty \\ T^n x \in K}} -\frac{1}{n}\log\mu(B_n(x, r)). \tag{2.4}$$

Proof of Theorem 1.3. This follows from Theorem 2.8, inequality (2.4) and Lemma 2.11. □

To end this subsection, for completeness of this section we give a relation between the upper δ-entropy and the upper local entropy in a general setting.

Theorem 2.12. *Let $T : X \to X$ be a continuous transformation of a metric space (X, d) and μ an ergodic T-invariant probability measure. Then for all $0 < \delta < 1$, we have*

$$\overline{h}_\mu^\delta(T) \leq \overline{h}_\mu^{loc}(T).$$

Proof. Fix $\varepsilon > 0$ and $0 < r < 1$. Define the set $X(\varepsilon, r, n') \subset X$, for $n' \geq 1$, by

$$X(\varepsilon, r, n') = \left\{ x \in X : -\frac{1}{n} \log \mu(B_n(x, r)) \leq \overline{h}_\mu^{loc}(T) + \varepsilon, \text{ for all } n \geq n' \right\}.$$

Note that $\mu(X(\varepsilon, r, n'))$ goes to 1 when $n' \to \infty$ and $r \to 0$. Take $r > 0$ small enough and $n'_0 = n'_0(r) > 0$ large enough such that $\mu(X(\varepsilon, r, n')) > 1 - \delta$ for every $n' \geq n'_0$. Let $K \subset X(\varepsilon, r, n'_0)$ be a compact set such that $\mu(K) > 1 - \delta$. We are going to find an upper bound of $S(n, r, K)$ (the maximal cardinality of a (n, r)-separated subset of K), for every $n \geq n'_0$. Let E be a maximal (n, r)-separated set in K. Since $(n, r/2)$-balls centered at E are disjoint, we have

$$\sum_{x \in E} \mu(B_n(x, r)) = \mu \left(\bigcup_{x \in E} B_n(x, r/2) \right) \leq 1.$$

As the (n, r)-balls with center in K satisfy $\mu(B_n(x, r)) \geq \exp\left(-n\left(\overline{h}_\mu^{loc}(T) + \varepsilon\right)\right)$, it follows that

$$\#E = S(n, r, K) \leq \exp\left(n\left(\overline{h}_m^{loc}(T) + \varepsilon\right)\right).$$

Therefore, by Lemma 2.2, we get

$$\begin{aligned} \limsup_{n\to\infty} \frac{1}{n} \log N(n, r, K) &\leq \limsup_{n\to\infty} \frac{1}{n} \log S(n, r, K) \\ &\leq \overline{h}_\mu^{loc}(T) + \varepsilon. \end{aligned}$$

Hence, for every $r > 0$ small enough we can find a compact $K \subset X$ such that $\mu(K) > 1 - \delta$ and

$$\limsup_{n\to\infty} \frac{1}{n} \log N(n, r, K) \leq \overline{h}_\mu^{loc}(T) + \varepsilon.$$

In particular,

$$\begin{aligned} \overline{h}_\mu^\delta(T) &= \lim_{r\to 0} \limsup_{n\to\infty} \frac{1}{n} \log N(n, r, \delta) \\ &\leq \lim_{r\to 0} \limsup_{n\to\infty} \frac{1}{n} \log N(n, r, K) \\ &\leq \overline{h}_\mu^{loc}(T) + \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, the conclusion follows. □

3. Exponential decay of the volume of dynamical balls

This section is divided in two parts. In the first part we define the Riemannian local entropy $h_\mu^{\mathcal{L}}(T)$ and we compare it with the classical measure entropy. In the second part, inspired by this property, we will show that the sum of positive Lyapunov exponents of a diffeomorphism $f : M \to M$ equals the exponential decay of the volume of dynamical balls for the dynamic of df on TM.

3.1. **A Riemannian local entropy for Riemannian manifolds.**

Our goal is to define a local entropy of a measure μ measuring the Riemannian measure $\mathcal{L}$ of a μ-typical dynamical ball. Moreover, we want to be able to compare it with the measure-theoretical entropy. It turns out that the essential supremum of the exponential decay for the Riemannian measure of μ-typical dynamical balls is an interesting quantity to consider.

Definition 3.1. Let $T : M \to M$ be a continuous transformation of a Riemannian manifold, preserving an ergodic T-invariant probability measure μ. For every compact set $K \subset M$ verifying $\mu(K) > 0$, we define the *local Riemannian entropy of T relative to μ over K*, denoted by $h_\mu^{\mathcal{L}}(T, K)$, as

$$h_\mu^{\mathcal{L}}(T, K) = \operatorname*{ess\,sup}_{x\in K} \lim_{r\to 0} \limsup_{\substack{n\to\infty \\ T^n x\in K}} -\frac{1}{n} \log \mathcal{L}(B_n(x, r)),$$

where the essential supremum is with respect to the measure μ. We define the *local Riemannian entropy of T* relative to μ, denoted by $h_\mu^{\mathcal{L}}(T)$, as

$$h_\mu^{\mathcal{L}}(T) = \sup_K h_\mu^{\mathcal{L}}(T, K),$$

where the supremum is taken over all the compact subsets K of M verifying $\mu(K) > 0$.

The following theorem shows that the lower δ-entropy of Katok is bounded from above by the local Riemannian entropy. We stress the fact that the measure μ only appears in the definition of local Riemannian entropy when considering μ-typical dynamical balls. For the best of our knowledge there are no related results in the literature.

Theorem 3.2. *Let $T : M \to M$ be a continuous transformation of a complete Riemannian manifold, preserving an ergodic T-invariant probability measure μ. If $K \subset M$ is a compact set of strictly positive μ-measure, then for all $1 - \mu(K)^2 < \delta < 1$, we have*

$$\underline{h}_\mu^\delta(T) \le h_\mu^{\mathcal{L}}(T, K).$$

Proof. If $h_\mu^{\mathcal{L}}(T, K) = \infty$ there is nothing to prove. Suppose that $h_\mu^{\mathcal{L}}(T, K) < \infty$. For $\varepsilon > 0$, $r > 0$ and $n' \ge 1$, we define the set $K_{\varepsilon,r,n'}$ as

$$K_{\varepsilon,r,n'} = \{x \in K \ : \ \mathcal{L}(B_n(x, r)) \ge \exp(-n(h_\mu^{\mathcal{L}}(T, K) + \varepsilon)), \text{ for every } n \ge n' \text{ such that } T^n x \in K\}.$$

Note that the measure $\mu(K_{\varepsilon,r,n'})$ goes to $\mu(K)$ when $n' \to \infty$ and $r \to 0$. For all $0 < \eta < \mu(K)/2$ there exist $r > 0$ and $n'_0 \ge 1$ (depending on r) such that $\mu(K_{\varepsilon,r,n'_0}) > \mu(K) - \eta/2$. Let $K_0 \subset K_{\varepsilon,r,n'_0}$ be a compact set with measure $\mu(K_0) > \mu(K) - \eta$. We are going to estimate the cardinality of a minimal (n, r)-covering of K_0 for $n \ge n'_0$. The problem is that in general, if $x, x' \in K$ are different, the first time of return in K is also different for these two points. The ergodicity assumption for the dynamical system will help us to erase this problem.

Birkhoff's Ergodic Theorem implies that $\frac{1}{n}\sum_{i=0}^{n-1}\mu(K_0\cap T^{-i}K_0)$ converge to $\mu(K_0)^2$. In particular, there is a sequence $(\phi(n))_n$ strictly increasing of integers such that $\mu(K_0\cap T^{-\phi(n)}K_0)$ converge to $L(K_0)\geq\mu(K_0)^2$. Let $0<\lambda<L(K_0)/2$. Then, there is an integer $n_1\geq 1$ such that $\mu(K_0\cap T^{-\phi(n)}K_0)>L(K_0)-\lambda$ for all $n\geq n_1$. Let $\delta(K_0,\lambda)=1-(\mu(K_0)^2-\lambda)$ and set $K_{\phi(n)}=K_0\cap T^{-\phi(n)}K_0$. The μ-measure of $K_{\phi(n)}$ satisfies, for all $n\geq n_1$

$$\mu(K_{\phi(n)})>L(K_0)-\lambda\geq\mu(K_0)^2-\lambda=1-\delta(K_0,\lambda).$$

Let E be a maximal set $(\phi(n),r)$-separated in $K_{\phi(n)}$, for $n\geq\max\{n_0,n_1\}$. Then

$$\begin{aligned}\mathcal{L}(V_r(K)) &\geq \mathcal{L}\left(\bigcup_{x\in E}B_{\phi(n)}(x,r/2)\right)\\ &\geq \sum_{x\in E}\mathcal{L}(B_{\phi(n)}(x,r/2))\\ &\geq \#E\exp(-n(h_\mu^{\mathcal{L}}(T,K)+\varepsilon)).\end{aligned}$$

Therefore, the cardinality of E is bounded from above by

$$\#E\leq\mathcal{L}(V_r(K))\exp(n(h_\mu^{\mathcal{L}}(T,K)+\varepsilon)).$$

Hence, using Lemma 2.2 and the estimation from above of the cardinality of a maximal set $(\phi(n),r)$-separated in $K_{\phi(n)}$, we get

$$\begin{aligned}\liminf_{n\to\infty}\frac{1}{n}\log N(n,r,\delta(K_0,\lambda)) &\leq \liminf_{n\to\infty}\frac{1}{\phi(n)}\log N(\phi(n),r,\delta(K_0,\lambda))\\ &\leq \liminf_{n\to\infty}\frac{1}{\phi(n)}\log N(\phi(n),r,K_{\phi(n)})\\ &\leq \liminf_{n\to\infty}\frac{1}{\phi(n)}\log S(\phi(n),r,K_{\phi(n)})\\ &\leq h_\mu^{\mathcal{L}}(T,K)+\varepsilon.\end{aligned}$$

In particular, we have shown that for every $r>0$ small enough and every n large enough (depending on r), there exists a compact set $K_{\phi(n)}$ of μ-measure $\mu(K_{\phi(n)})\geq\delta(K_0,\lambda)$. Therefore, the sequence of inequalities above implies that

$$\begin{aligned}\underline{h}_\mu^{\delta(K_0,\lambda)}(T) &= \lim_{r\to 0}\liminf_{n\to\infty}\frac{1}{n}\log N(n,r,\delta(K_0,\lambda))\\ &\leq h_\mu^{\mathcal{L}}(T,K)+\varepsilon.\end{aligned}$$

Since $\lambda>0$ is arbitrary and from Proposition 2.4, we have $\underline{h}_\mu^{\delta}(T)\leq h_\mu^{\mathcal{L}}(T,K)+\varepsilon$ for all $1-\mu(K_0)^2<\delta<1$. Since $\eta>0$ is arbitrary, we have $\underline{h}_\mu^{\delta}(T)\leq h_\mu^{\mathcal{L}}(T,K)+\varepsilon$, for all $1-\mu(K)^2<\delta<1$. Since $\varepsilon>0$ is arbitrary, the conclusion of the theorem follows. □

A direct consequence of Theorem 3.2, by choosing a sequence of compact sets $(K_n)_n$ such that $\mu(K_n)\to 1$ when $n\to 1$, is the following corollary.

Corollary 3.3. *Let $T : M \to M$ be a continuous transformation of a complete Riemannian manifold, preserving an ergodic T-invariant probability measure μ. Then, for all $0 < \delta < 1$, we have*

$$\underline{h}^{\delta}_{\mu}(T) \leq h^{\mathcal{L}}_{\mu}(T).$$

Finally we can prove Theorem 1.4.

Proof of Theorem 1.4. This is a consequence of Theorem 2.5 and Corollary 3.3. □

We have considered the Riemannian measure in the definition of local Riemannian entropy because we can ensure always that it gives positive measure to μ-typical dynamical balls, regardless of the measure μ. Moreover, it will be very useful in the proof of Theorem 1.1. However, we might also have considered the measure μ as in the case of local entropies, and nothing in the proof of Theorem 1.4 changes. Hence, for all $0 < \delta < 1$, we have

$$\underline{h}^{\delta}_{\mu}(T) \leq \sup_{K} \operatorname*{ess\,sup}_{x\in K} \lim_{r\to 0} \limsup_{\substack{n\to\infty \\ T^n x\in K}} -\frac{1}{n}\log \mu(B_n(x,r)). \tag{3.1}$$

Theorem 3.4. *Let $T : M \to M$ be a Lipschitz transformation of a complete Riemannian manifold and μ an ergodic T-invariant probability measure. Then, for all $0 < \delta < 1$, we have*

$$h_{\mu}(T) = \underline{h}^{\delta}_{\mu}(T).$$

Proof. On the one hand, Theorem 2.5 implies that $h_{\mu}(T) \leq \underline{h}^{\delta}_{\mu}(T)$. On the other hand, Lemma 2.11 (or Theorem 1.3) imply that $\underline{h}^{\delta}_{\mu}(T) \leq h_{\mu}(T)$. □

We now can now summarize Theorems 1.3, 2.9 and 3.4 in one single statement.

Corollary 3.5. *Let $T : M \to M$ be a Lipschitz transformation of a complete Riemannian manifold and μ an ergodic T-invariant probability measure. Then*

$$h_{\mu}(T) = \underline{h}^{\delta}_{\mu}(T) = \underline{h}^{loc}_{\mu}(T) = \sup_{K} \operatorname*{ess\,sup}_{x\in K} \lim_{r\to 0} \limsup_{\substack{n\to\infty \\ T^n x\in K}} -\frac{1}{n}\log \mu(B_n(x,r))$$

3.2. **Lyapunov exponents and exponential decay of linearized dynamical balls.** Now we will study the dynamics of a smooth transformation of a noncompact Riemannian manifold. Let (M, g) be a Riemannian manifold and $f : M \to M$ a C^1-diffeomorphism. For $x \in M$, let $\|\cdot\|_x$ denote the Riemannian norm induced by g on T_xM. The point x is said to be (Lyapunov-Perron) *regular* if there exist numbers $\{\lambda_i(x)\}_{i=1}^{l(x)}$, called *Lyapunov exponents*, and a decomposition of the tangent space at x into $T_xM = \bigoplus_{i=1}^{l(x)} E_i(x)$ such that for every tangent vector $v \in E_i(x) \setminus \{0\}$, we have

$$\lim_{n\to\pm\infty} \frac{1}{n}\log \|d_x f^n v\|_{f^n x} = \lambda_i(x),$$

and

$$\lim_{n\to\pm\infty} \frac{1}{n}\log |\det(d_x f^n)| = \sum_{i=1}^{l(x)} \lambda_i(x)\dim(E_i(x)).$$

Let Λ be the set of regular points. By Oseledec's Theorem ([Ose68], [Led84b]), if μ is an f-invariant probability measure on M such that $\log \|df^{\pm 1}\|$ is μ-integrable, then the set Λ has full μ-measure. Moreover, the functions $x \mapsto \lambda_j(x)$ and $x \mapsto \dim(E_j(x))$ are μ-measurable and f-invariant. In particular, if μ is ergodic, they are μ-almost everywhere constant. In that case, we denote by $\{\lambda_j\}_{j=1}^{l}$ the Lyapunov exponents. Note that for μ-a.e. $x \in M$, for every $v \in T_xM \setminus \{0\}$ the limit $\lim_{n\to\pm\infty} \frac{1}{n} \log \|d_x f^n(v)\|_{f^n x}$ exists. More precisely, if $v = \sum_j v_j$ is the Oseledec's decomposition of v, then

$$\lim_{n\to\pm\infty} \frac{1}{n} \log \|d_x f^n(v)\|_{f^n x} =: \lambda(x,v)$$

is the largest Lyapunov exponent associated to a vector $v_j \neq 0$.
Let $x \in \Lambda$. Define $E^{su}(x)$ as

$$E^{su}(x) = \bigoplus_{\lambda_j(x)>0} E_j(x).$$

Consider the function $\chi^+ : M \to \mathbb{R}$ defined as $\chi^+(x) = \sum_{\lambda_j(x)>0} \lambda_j(x) \dim(E_j(x))$ if $x \in \Lambda$ and $\chi^+(x) = 0$ otherwise. If μ is an ergodic f-invariant probability measure on M, we denote by $\chi^+(\mu)$ (or simply χ^+ when there is no ambiguity) the essential value of the function $\chi^+(x)$ with respect to μ.

Denote by $B_x(0,r)$ the r-ball centered at 0 in (T_xM, g_x).

Definition 3.6. We define the *tangent* (n,r)*-dynamical ball* $\mathcal{C}(x,n,r)$ on T_xM as

$$\mathcal{C}(x,n,r) = \bigcap_{i=0}^{n-1} (d_x f^i)^{-1}(B_{f^i x}(0,r)) \subset T_xM.$$

Let vol_x be the Euclidean volume on T_xM induced by g_x. Theorem 3.7 below says that the Lyapunov exponents describe the exponential decay of the vol_x-volume of a tangent dynamical ball, as follows

Theorem 3.7. *Let (M,g) be a Riemannian manifold and $f : M \to M$ a C^1-diffeomorphism. Suppose that μ is an f-invariant probability measure such that $\log \|df^{\pm 1}\| \in L^1(\mu)$. Then, for μ-a.e. $x \in M$, we have*

$$\begin{aligned} \lim_{r\to 0} \liminf_{n\to\infty} -\frac{1}{n} \log vol_x(\mathcal{C}(x,n,r)) &= \lim_{r\to 0} \limsup_{n\to\infty} -\frac{1}{n} \log vol_x(\mathcal{C}(x,n,r)) \\ &= \chi^+(x). \end{aligned}$$

Before giving a proof of Theorem 3.7, we need the following technical lemma.

Lemma 3.8. *Let (M,g) be a Riemannian manifold of dimension d and $f : M \to M$ a C^1-diffeomorphism. Suppose that μ is an ergodic f-invariant probability measure such that $\log \|df^{\pm 1}\| \in L^1(\mu)$. Then, for every $\varepsilon > 0$, there exists a compact set $K \subset M$ such that for every $x \in K$ there is a sequence $(t_n)_n$ of strictly positive real numbers such that*

1. *the μ-measure of K is greater than $1-\varepsilon$,*
2. *the exponential decay of t_n is smaller than 2ε, i.e.*

$$\limsup -\frac{1}{n} \log t_n \leq 2\varepsilon,$$

3. *for every $r > 0$ there exist constants $C, C' > 0$ (depending on K, r and l) such that, for every integer $n \geq 0$ and every $x \in K$, we have*

$$vol_x(\mathcal{C}(x,n,r)) \geq C t_n^d \prod_{\lambda_j>0} \exp(-n \dim(E_j(x))(\lambda_j + \varepsilon)) \tag{3.2}$$

and

$$vol_x(\mathcal{C}(x,n,r)) \le C' \prod_{\lambda_j>0} \exp(-n\dim(E_j(x))(\lambda_j-\varepsilon)). \tag{3.3}$$

Proof. Let $\varepsilon > 0$ be such that $\varepsilon \le \min\{|\lambda_j| : \lambda_j \ne 0\}/100$. For every integer $k \ge 1$, define the set $M_{\varepsilon,k}$ as

$$\begin{aligned} M_{\varepsilon,k} = \{x \in \Lambda : \quad & \forall v \in T_x^1 M,\ \forall |i| \ge k, \\ & \exp(i(\lambda(x,v)-\varepsilon)) \le \|d_x f^i v\| \le \exp(i(\lambda(x,v)+\varepsilon))\}. \end{aligned}$$

Oseledec's Theorem implies that $\mu(M \setminus M_{\varepsilon,k})$ goes to 0 when k goes to infinity. In particular, there exists $k_0 \ge 1$ such that $\mu(M \setminus M_{\varepsilon,k_0}) \le \varepsilon/2$. Since μ is a Borel measure, there is a compact $K \subset M_{\varepsilon,k_0}$ such that $\mu(K) > 1-\varepsilon$, $\int_{M\setminus K} \log^+ \|df^{\pm 1}\| d\mu < \varepsilon$ and the maps $x \mapsto E_j(x)$ are continuous over K. For $x \in K$ and $n \ge 0$, define the sets of integers $I_{x,n}$ and $I^c_{x,n}$ as

$$I_{x,n} = \{k_0 \le i \le n : f^i x \in K\}$$

and

$$I^c_{x,n} = \{0 \le i \le n : f^i x \notin K\}.$$

From the definition of a tangent dynamical ball, we have

$$\mathcal{C}(x,n,r) = \mathcal{C}_K(x,n,r) \cap \mathcal{C}_{K^c}(x,n,r),$$

where

$$\begin{aligned} \mathcal{C}_K(x,n,r) &= \bigcap_{i\in I_{x,n}} (d_{f^i x} f^{-i})(B_{f^i x}(0,r)), \quad \text{and} \\ \mathcal{C}_{K^c}(x,n,r) &= \bigcap_{i\in I^c_{x,n}} (d_{f^i x} f^{-i})(B_{f^i x}(0,r)). \end{aligned}$$

To estimate the volume $\mathrm{vol}_x(\mathcal{C}(x,n,r))$ we will use that

$$(d_x f^i)(B_x(0,r)) \subset B_{f^i x}(0, \|d_x f^i\| r) \subset T_{f^i x} M, \tag{3.4}$$

and

$$B_x(0, \|d_x f^i\|^{-1} r) \subset (d_{f^i x} f^{-i})(B_{f^i x}(0,r)). \tag{3.5}$$

By continuity of the maps $x \mapsto E_j(x)$ on K, we can assume that there exists $\alpha > 0$ such that for every $x \in K$ and every pair (j_1, j_2), with $j_1 \ne j_2$, we have

$$\angle(E_{j_1}(x), E_{j_2}(x)) \ge \alpha. \tag{3.6}$$

Part I: Upper bound of the volume. Inequality (3.3) follows directly from $\mathcal{C}(x,n,r) \subset \mathcal{C}_K(x,n,r)$. Let $w = (d_{f^i x} f^{-i})v$, where $v \in B_{f^i x}(0,r) \cap E_j(f^i x)$, $1 \le j \le s$ and $i \in I_{x,n}$. Using (3.4), we have

$$\begin{aligned} \|w\| = \|d_{f^i x} f^{-i} v\| &\le \exp(-i(\lambda(f^i x, v) - \varepsilon))\|v\| \\ &= \exp(-i(\lambda_j - \varepsilon))\|v\|. \end{aligned}$$

Let $v \in B_z(0,r)$, for $z \in K$, and let $v = \sum_j v_j$ be the Oseledec's decomposition of v. The *law of sines* implies

$$\|v_j\|_z \le \frac{\|v\|_z}{\sin(\alpha)}.$$

Therefore,

$$\begin{aligned}(d_{f^ix}f^{-i})(B_{f^ix}(0,r)) &\subseteq (d_{f^ix}f^{-i})\left(\prod_{j=1}^{l} B^j_{f^ix}(0,r/\sin(\alpha))\right)\\ &\subseteq \prod_{j=1}^{l}(d_{f^ix}f^{-i})(B^j_{f^ix}(0,r/\sin(\alpha)))\\ &\subseteq \left(\prod_{\lambda_j\leq 0} B^j(0,r/\sin(\alpha))\right)\times\\ &\quad \left(\prod_{\lambda_j>0} B^j(0,\exp(-i(\lambda_j-\varepsilon))r/\sin(\alpha))\right).\end{aligned}$$

The last term above is a parallelepiped (of dimension d). His volume is comparable to the volume of a rectangular parallelepiped whose sides have the same lengths that the original one. The constant of comparison depends only on the angle of the sides, which is greater than α by (3.6). Therefore, there exists a constant $\tilde{C}'=\tilde{C}'(\alpha)>0$, such that

$$\mathrm{vol}_x(\mathcal{C}(x,n,r))\leq \tilde{C}'(r/\sin(\alpha))^d\prod_{\lambda_j>0}\exp(-n\dim(E_j(x))(\lambda_j-\varepsilon)).$$

In particular, Inequality (3.3) is deduced for $C'=\tilde{C}'(r/\sin(\alpha))^d$.

Part II: Lower bound of the volume. As we have no control on the behavior of the differential on $M\setminus K$, we need to reduce the estimation problem of the volume of $\mathcal{C}(x,n,r)$ to a problem of estimation of the volume of $\mathcal{C}_K(x,n,r)$. For $i\in I_{x,n}$, define $j(i)$ as the number of consecutive indices greater than i belonging to $I^c_{x,n}$, that is,

$$j(i)=\begin{cases}\max\{j\geq 1: i+m\in I^c_{x,n},\forall 1\leq m\leq j\}, & \text{if } i+1\in I^c_{x,n}\\ 0, & \text{otherwise.}\end{cases}$$

Let $i\in I_{x,n}$ and suppose $j(i)\geq 1$. By definition, we have $i+m\in I^c_{x,n}$ for every $1\leq m\leq j(i)$ and $i+j(i)+1\in I_{x,n}$. Using inclusion (3.5), it follows

$$\begin{aligned}d_{f^{i+m}x}f^{-(i+m)}(B_{f^{i+m}x}(0,r)) &= d_{f^ix}f^{-i}d_{f^{i+m}x}f^{-m}(B_{f^{i+m}x}(0,r))\\ &\supset d_{f^ix}f^{-i}B_{f^ix}(0,\|d_{f^ix}f^m\|^{-1}r)\\ &\supset d_{f^ix}f^{-i}B_{f^ix}\left(0,\left(\prod_{m'=0}^{m-1}\min\{1,\|d_{f^{i+m'}x}f\|^{-1}\}\right)r\right)\\ &\supset d_{f^ix}f^{-i}B_{f^ix}\left(0,\left(\prod_{k\in I^c_{x,n}}\min\{1,\|d_{f^kx}f\|^{-1}\}\right)r\right).\end{aligned}$$

The above sequence of inclusions implies that every set of the form $d_{f^kx}f^{-k}(B_{f^kx}(0,r))$, for $k\in I^c_{x,n}$, contains a set of the form $(d_{f^ix}f^{-i})(B_{f^ix}(0,t_nr))$, where $i\in I_{x,n}\cup\{0\}$

and $t_n = \prod_{k \in I^c_{x,n}} \min\{1, \|d_{f^{k-1}x} f\|^{-1}\}$. Thus,

$$\mathcal{C}(x,n,r) \supset \mathcal{C}_K(x,n,t_n r). \tag{3.7}$$

Claim (2) follows from Birkhoff's Ergodic Theorem. More precisely,

$$\begin{aligned}
-\frac{1}{n}\log t_n &= -\frac{1}{n}\sum_{k \in I^c_{x,n}} \log\min\{1, \|d_{f^k x} f\|^{-1}\} \\
&\leq -\frac{1}{n}\sum_{k=0}^{n-1} \mathbf{1}_{K^c}(f^k x)\log\min\{1, \|d_{f^k x} f\|^{-1}\} \\
&\leq \int_{M\setminus K} \log^+ \|df\| d\mu + \varepsilon \\
&\leq 2\varepsilon
\end{aligned}$$

for every n large enough.

Let $i \in I_{x,n}$. For $w = (d_{f^i x} f^{-i})v$, where $v \in B_{f^i x}(0,r) \cap E_j(f^i x)$ and $1 \leq j \leq l$, we have $w \in E_j(x)$. In particular, by definition of M_{ε,k_0}, we obtain for $i \geq k_0$ the following

$$\begin{aligned}
\|w\| = \|d_{f^i x} f^{-i} v\| &\geq \exp(-i(\lambda(f^i x, v) + \varepsilon))\|v\| \\
&= \exp(-i(\lambda_j + \varepsilon))\|v\|.
\end{aligned} \tag{3.8}$$

Consider now $B^j_x(0,r)$ the r-ball centered at 0 on $E_j(x)$, for the norm $g_x|_{E_j(x)}$, for all $1 \leq j \leq l$. Thus, using (3.8), it follows

$$\begin{aligned}
(d_{f^i x} f^{-i})(B_{f^i x}(0,r)) &\supseteq (d_{f^i x} f^{-i})\left(\prod_{j=1}^{l} B^j_{f^i x}(0, r/l)\right) \\
&= \prod_{j=1}^{l} (d_{f^i x} f^{-i})(B^j_{f^i x}(0,r/l)) \\
&\supset \prod_{j=1}^{l} B^j_x(0, \exp(-i(\lambda_j+\varepsilon))r/l) \\
&\supset \left(\prod_{\lambda_j \leq 0} B^j_x(0,r/l)\right) \times \left(\prod_{\lambda_j > 0} B^j_x(0, \exp(-i(\lambda_j+\varepsilon))r/l)\right).
\end{aligned}$$

The same arguments of Part I for the upper bound of the volume allow to show the existence of a constant $\tilde{C} = \tilde{C}(\alpha) > 0$ such that

$$vol_x\left(\prod_{\lambda_j \leq 0} B^j_x(0,r/l) \times \prod_{\lambda_j > 0} B^j_x(0, \exp(-i(\lambda_j+\varepsilon))r/l)\right)$$

is greater than

$$\tilde{C}(r/l)^d \prod_{\lambda_j > 0} \exp(-n\dim(E_j(x))(\lambda_j + \varepsilon)).$$

The inequality above together (3.7) implies (3.2) for $C = \tilde{C}(r/l)^d$. It concludes the proof of Lemma 3.8. □

We will use Lemma 3.8 to prove Theorem 3.7 for ergodic measures. This is sufficient because of the ergodic decomposition of a measure.

Proof of Theorem 3.7. Suppose that μ is an ergodic measure and let $\varepsilon > 0$ be small enough as in Lemma 3.8. Let $K = K(\varepsilon, k_0)$ be the compact given by Lemma 3.8. Then, for every $x \in K$, we have

$$\begin{aligned}\lim_{r\to 0}\limsup_{n\to\infty} -\frac{1}{n}\log \mathrm{vol}_x(\mathcal{C}(x,n,r)) &\leq \limsup_{n\to\infty} -\frac{1}{n}\log t_n + \sum_{\lambda_j>0}\dim(E_j(x))(\lambda_j+\varepsilon)\\ &\leq 2d\varepsilon + \sum_{\lambda_j>0}\dim(E_j(x))(\lambda_j+\varepsilon)\end{aligned}$$

and

$$\lim_{r\to 0}\liminf_{n\to\infty} -\frac{1}{n}\log \mathrm{vol}_x(\mathcal{C}(x,n,r)) \geq \sum_{\lambda_j>0}\dim(E_j(x))(\lambda_j-\varepsilon).$$

Since we can choose $K = K(\varepsilon)$ such that $\mu(K) \to 1$ when $\varepsilon \to 0$, the conclusion of Theorem 3.7 follows. □

Let $x \in M$. Denote by $r_{inj}(x)$ the injectivity radius at x. For $0 < r \leq r_{inj}(x)$ we define the *linearized* (n,r)*-dynamical ball* as $\mathcal{C}_n(x,r) = \exp_x(\mathcal{C}(x,n,r))$. Observe that $(\exp_x)_*\mathrm{vol}_x$ is locally comparable with the Riemannian measure $\mathcal{L}$, therefore Theorem 3.7 implies the following:

Corollary 3.9. *Let M be a Riemannian manifold and let $f : M \to M$ be a C^1-diffeomorphism. Suppose that μ is an f-invariant probability measure such that* $\log\|df^{\pm 1}\| \in L^1(\mu)$. *Then, for μ-a.e. $x \in M$, we have*

$$\begin{aligned}\lim_{r\to 0}\liminf_{n\to\infty} -\frac{1}{n}\log \mathcal{L}(\mathcal{C}_n(x,r)) &= \lim_{r\to 0}\limsup_{n\to\infty} -\frac{1}{n}\log \mathcal{L}(\mathcal{C}_n(x,r))\\ &= \chi^+(x).\end{aligned}$$

Theorem 1.4 and Corollary 3.9 are a first step in order to prove Ruelle's inequality for "nice" diffeomorphisms of noncompact manifolds. Heuristically speaking, if a dynamical ball $B_n(x,r)$ is comparable with the linearized dynamical ball $\mathcal{C}_n(x,r)$, then Ruelle's inequality should arise in a natural way since the limits in Corollary 3.9 look like the Riemannian local entropy.

4. Geodesic flow in negative curvature

Let X be a complete Riemannian manifold with dimension at least 2 and pinched negative sectional curvature at most -1. Let T^1X his unit tangent bundle. Recall that the Liouville measure $\mathcal{L}$ on T^1X is the Riemannian volume of the Sasaki metric on T^1X (see for instance [Bal95] for details). It is invariant under the action of the geodesic flow (g^t) on T^1X. Let $v \in T^1X$ and $t \in \mathbb{R}$. Denote by $E^{su}(v)$ the tangent space of the strong unstable manifold at v. Denote by $J^{su}(v,t)$ the Jacobian of the linear map $d_vg^t|_{E^{su}(v)}$. The *geometric potential* $F^{su} : T^1X \to \mathbb{R}$ is then defined by

$$F^{su}(v) = -\frac{d}{dt}_{|t=0}\log J^{su}(v,t).$$

Theorem 4.1 (Paulin-Pollicott-Schapira)**.** *Let X be a complete Riemannian manifold with dimension at least 2 and pinched negative sectional curvature at most -1. Assume that the derivatives of the sectional curvature are uniformly bounded. Then F^{su} is Hölder-continuous and bounded.*

The potential F^{su} is intimately related to the Lyapunov exponents. Let μ be a probability measure on T^1X invariant under the geodesic flow (g^t). Since the sectional curvature is pinched, the norm $\|dg^{\pm 1}\|$ is bounded. Hence, $\log \|dg^{\pm 1}\|$ is μ-integrable. Oseledec's Theorem implies that μ-almost every $v \in T^1X$ is regular. In particular, for μ-almost every $v \in T^1X$, the tangent space of the strong unstable manifold at v coincides with $\bigoplus_{\lambda_j(v)>0} E_j(v)$. This fact justifies the notation $E^{su}(v)$ for the direct sum of the spaces $E_j(v)$ associated to $\lambda_j(v) > 0$. Moreover, we have

$$\lim_{n\to+\infty} \frac{1}{n} \int_0^n F^{su}(g^t v)dt = -\lim_{n\to+\infty} \frac{1}{n} \log J^{su}(v,n) = -\chi^+(v), \quad \mu-\text{a.e.}$$

The key fact that will allow us to prove Ruelle's inequality for the geodesic flow is the Gibbs property of the Liouville measure for the potential F^{su} ([PPS12, Proposition 7.9]). Recall that a (g^t)-invariant measure m on T^1X satisfies the Gibbs property for the potential $F: T^1X \to \mathbb{R}$ with constant $c(F)$ if and only if for every compact subset K of T^1X, for every $r > 0$, there exists $C = C(K,r) \geq 1$, such that for every $T \geq 0$, for every $v \in K \cap g^{-T}K$, we have

$$C^{-1} \leq \frac{m(B_T(v,r))}{\exp(\int_0^T (F(g^t v) - c(F))dt)} \leq C.$$

Proposition 4.2 (Paulin-Pollicott-Schapira)**.** *Let X be a complete Riemannian manifold with dimension at least 2 and pinched negative sectional curvature at most -1. Assume that the derivatives of the sectional curvature are uniformly bounded. Then the Liouville measure on T^1X satisfies the Gibbs property for the potential F^{su} and the constant $c(F^{su}) = 0$.*

Note that the assumption on the derivatives of the sectional curvature is crucial. It implies in particular that the strong unstable and strong stable distributions are Hölder-continuous (see for instance [PPS12, Theorem 7.3]), and therefore that $\mathcal{L}$ locally decomposes into a product of Lebesgue measures along unstable and stable manifolds (see [PPS12, Theorem 7.6]). This last fact is the cornerstone in order to estimate the Liouville measure of a dynamical ball.

4.1. **Ruelle's inequality and Pesin's formula.** Let X be a complete Riemannian manifold satisfying the assumptions of Theorems 1.1 and 1.2. By simplicity we will always consider in the proofs an ergodic (g^t)-invariant probability measure μ. The proofs of the theorems in the non-ergodic case are consequence of the ergodic decomposition of such a measure. We can also assume that $g = g^1$ is an ergodic transformation with respect to μ. If it is not the case, then we can choose an ergodic-time τ for μ (see [LS79, Theorem 3.2]) and prove Theorem 1.1 and Theorem 1.2 for the map g^τ. The validity of Theorems 1.1 and 1.2 for g^τ implies the validity of both theorems for g since $h_\mu(g^\tau) = \tau h_\mu(g)$ and the Lyapunov exponents of g^τ are τ-multiples of the Lyapunov exponents for g.

Proof of Theorem 1.1. Let K be a subset of T^1X of measure $0<\mu(K)<1$. Since F^{su} is μ-integrable, Proposition 4.2 implies

$$\begin{aligned} h_\mu^{\mathcal{L}}(g,K) &= \operatorname*{ess\,sup}_{v\in K} \lim_{r\to 0} \limsup_{\substack{n\to\infty \\ g^n v\in K}} -\frac{1}{n}\log \mathcal{L}(B_n(v,r)) \\ &\leq \operatorname*{ess\,sup}_{v\in K} \lim_{r\to 0} \limsup_{\substack{n\to\infty \\ g^n v\in K}} -\frac{1}{n}\log\left(C^{-1}\exp\left(\int_0^n F^{su}(g^t v)dt\right)\right) \\ &= -\int F^{su}d\mu \\ &= \chi^+(\mu). \end{aligned}$$

Last equality is a consequence of Birkhoff's Ergodic Theorem. Thus, $h_\mu^{\mathcal{L}}(g)\leq \chi^+$ and Theorem 1.4 implies directly that

$$h_\mu(g)\leq \chi^+.$$

□

The proof of Theorem 1.2 is similar to those in [LS82], [Led84a] and [LY85]. We only need to corroborate that all technical hypotheses hold, for instance the Hölder regularity of strong unstable and strong stable distributions. As said before, these technical hypotheses are consequence of the assumption on the derivatives of the sectional curvature. In [OP04] the authors use the regularity of the strong unstable foliation to prove the existence of nice measurable partitions. They follow the ideas in [LS82] and [Led84a] adapted to the geodesic flow in negative curvature. We stress that in [OP04] the authors use the Hölder regularity of strong unstable and strong stable foliations omitting the hypothesis on the derivatives of the sectional curvatures, even if it is necessary to ensure such regularity (we refer to [BBB87] where the authors construct a finite volume Riemannian surface with pinched negative sectional curvatures whose strong stable foliation is not Hölder-continuous).

Recall that a measurable partition ξ of T^1X is subordinate to the W^{su}-foliation if for μ-a.e. $v\in T^1X$, we have

(i) the atom $\xi(v)$ is contained in $W^{su}(v)$, and
(ii) the atom $\xi(v)$ contains a neighborhood of v, open for the submanifold topology on $W^{su}(v)$.

Let vol_v be the volume on $W^{su}(v)$ induced by the Sasaki metric on T^1X restricted to the strong unstable manifold $W^{su}(v)$. The measure μ has absolutely continuous conditional measures on unstable manifolds if for every μ-measurable partition ξ subordinate to W^{su}, the conditional measure $\mu_{\xi(v)}$ of μ on $\xi(v)$ is absolutely continuous with respect to vol_v.

Proposition 4.3. *Let X be a complete Riemannian manifold with dimension at least 2 and pinched negative sectional curvature at most -1. Assume that the derivatives of the sectional curvature are uniformly bounded. Let μ an ergodic (g^t)-invariant probability measure and suppose that $g=g^1$ is ergodic. Then, there exists a μ-measurable partition ξ of T^1X, such that*

(1) *the partition ξ is decreasing, i.e. $(g^{-1}\xi)(v)\subset\xi(v)$ for μ-a.e. $v\in T^1X$,*
(2) *the partition $\bigvee_{n\geq 0} g^{-n}\xi$ is the partition into points,*
(3) *the partition ξ is subordinate to the W^{su}-foliation,*

(4) *for μ-a.e. v, we have $\bigcup_{n\in\mathbb{Z}} g^n\xi(g^n v) = W^{su}(v)$,*
(5) *for all measurable sets $B \subset T^1X$, the map*

$$\psi_B(v) = \mathrm{vol}_v(\xi(v) \cap B)$$

is measurable and μ-a.e. finite,
(6) *for μ-a.e. $v \in T^1X$, if $w, w' \in \xi(v)$, then the infinite product*

$$\Delta(w,w') = \frac{\prod_{n=0}^{\infty} J^{su}(g^{-n}w,1)}{\prod_{n=0}^{\infty} J^{su}(g^{-n}w',1)}$$

converges, and
(7) *there exist constants $C > 0$ and $0 < \alpha < 1$ such that, if $w \in \xi(v)$, then*

$$|\log \Delta(v,w)| \leq C(d(v,w))^{\alpha}.$$

Proof. The existence of μ-measurable partitions satisfying $(1) - (4)$ is proved in [OP04]. Properties $(5) - (7)$ are consequence of the regularity of the strong unstable distribution and the regularity of J^{su}, following the same proof of [Led84a, Proposition 3.1]. □

The class of μ-measurable partitions satisfying $(1)-(4)$ contains somehow all the complexity of the dynamics of the geodesic flow in the sense that every partition in this class maximises the measure-theoretic entropy. This result is proved in [OP04] following the ideas in [Led84a] and [LY85].

Proposition 4.4 (Ledrappier-Young/Otal-Peigné)**.** *Let X be a complete Riemannian manifold with dimension at least 2 and pinched negative sectional curvature at most -1. Assume that the derivatives of the sectional curvature are uniformly bounded. Let μ an ergodic (g^t)-invariant probability measure and suppose that $g = g^1$ is ergodic. If ξ is a partition as in Proposition 4.3, then*

$$h_\mu(g) = h_\mu(g,\xi).$$

Proof of Theorem 1.2. We remark that the computation of the entropy appears in [LS82], but as this fact is not stated explicitly, we give the general idea behind. Suppose that μ has absolutely continuous conditional measures on unstable manifolds. Let ξ be a μ-measurable partition as in Proposition 4.3. We only have to prove that $h_\mu(g,\xi) = \chi^+$. This is equivalent to show that $H_\mu(g^{-1}\xi|\xi) = \int \log J^{su}(v,1)d\mu(v)$. Define the measure ν on T^1X by

$$\nu(B) = \int \mathrm{vol}_w(\xi(w) \cap B)d\mu(w),$$

for every measurable subset B of T^1X. Property (5) in Proposition 4.3 implies that ν is σ-finite. Since $\mu_{\xi(v)}$ is absolutely continuous with respect to vol_v, the measure μ is absolutely continuous with respect to ν. Moreover, the Radon-Nikodym derivative $\kappa = d\mu/d\nu$ coincide with $d\mu_{\xi(v)}/d\mathrm{vol}_v$, vol_v-almost everywhere on $\xi(v)$, for μ-almost every $v \in T^1X$ (see [LS82, Proposition 4.1]).
Recall that

$$H_\mu(g^{-1}\xi|\xi) = \int I_\mu(g^{-1}\xi|\xi)d\mu,$$

where $I_\mu(g^{-1}\xi|\xi)(v) = -\log \mu_{\xi(v)}((g^{-1}\xi)(v))$. Thus,

$$I_\mu(g^{-1}\xi|\xi)(v) = -\log \int_{(g^{-1}\xi)(v)} \kappa(w)d\mathrm{vol}_v(w).$$

Using Change of Variables Theorem, it follows

$$\int_{(g^{-1}\xi)(v)} \kappa(w) d\mathrm{vol}_v(w) = \int_{\xi(gv)} \kappa(g^{-1}w) \frac{1}{J^{su}(g^{-1}w,1)} d\mathrm{vol}_{gv}(w).$$

From [LS82, Proposition 4.2], the application $L(w) = \frac{\kappa(w)}{\kappa(g^{-1}w)} J^{su}(g^{-1}w,1)$ is constant on the atoms of the partition ξ. Therefore,

$$\begin{aligned}\int_{\xi(gv)} \kappa(g^{-1}w) \frac{1}{J^{su}(g^{-1}w,1)} d\mathrm{vol}_{gv}(w) &= \int_{\xi(gv)} \frac{\kappa(w)}{L(w)} d\mathrm{vol}_{gv}(w) \\ &= \frac{1}{L(gv)} \int_{\xi(gv)} \kappa(w) d\mathrm{vol}_{gv}(w) \\ &= \frac{1}{L(gv)} \int_{\xi(gv)} d\mu_{\xi(gv)}(w) \\ &= \frac{1}{L(gv)}.\end{aligned}$$

Putting all together, we have shown that $I_\mu(g^{-1}\xi|\xi) = \log J^{su}(v,1) + \log \frac{\kappa(gv)}{\kappa(v)}$. Since $I_\mu(g^{-1}\xi|\xi) \geq 0$ and $\log J^{su}(v,1)$ is μ-integrable, it follows that the negative part of $\log \frac{\kappa(gv)}{\kappa(v)}$ is μ-integrable. In particular, its μ-integral is equal to zero (see [LS82, Proposition 2.2]), thus

$$h_\mu(g) = \int I_\mu(g^{-1}\xi|\xi) d\mu = \int \log J^{su}(v,1) d\mu(v) = \chi^+.$$

The converse statement is just the conclusion of [Led84a, Theorem 3.4] under the hypothesis obtained in Proposition 4.4, for a μ-measurable partition ξ as in Proposition 4.3. □

4.2. **Further comments.** We discuss now some consequences of Theorem 1.1 in thermodynamic formalism. The *topological pressure of* (g^t) *for a potential* $F : T^1X \to \mathbb{R}$, denoted by $P_{(g^t)}(F)$ (or simply $P(F)$), is defined as

$$P(F) = \sup_\mu P(F,\mu),$$

where $P(F,\mu) = h_\mu((g^t)) + \int_{T^1X} F d\mu$ and μ is an (g^t)-invariant probability measure on T^1X. An (g^t)-invariant probability measure m on T^1X is said to be an *equilibrium state for* F, if

$$P(F) = P(F,m).$$

In [PPS12] the authors construct a Gibbs measure for every bounded Hölder-continuous potential F, with constant $c(F)$ equal to the topological pressure $P(F)$. We remark that if a Gibbs measure is finite, its normalization is an equilibrium state for the potential.

As a consequence of Theorem 4.1, there exists a Gibbs measure for F^{su} under the hypotheses of Theorem 1.1, which is denoted by $m_{F^{su}}$. In terms of thermodynamical formalism, Ruelle's inequality can be stated as:

Corollary 4.5. *Let X be a complete Riemannian manifold with dimension at least 2 and pinched negative sectional curvature at most -1. Assume that the derivatives of the sectional curvature are uniformly bounded. Then, for every (g^t)-invariant probability measure μ on T^1X, we have*

$$P(F^{su},\mu) \leq 0. \tag{4.1}$$

In particular, we can remove inequality (4.1) as a redundant assumption in [PPS12, Theorem 7.2]. Recall that the geodesic flow is conservative with respect to a finite or infinite measure m on T^1X if every wandering set has m-measure zero.

Corollary 4.6. *Let X be a complete Riemannian manifold with dimension at least 2 and pinched negative sectional curvature at most -1. Assume that the derivatives of the sectional curvature are uniformly bounded. If the geodesic flow on T^1X is conservative with respect to the Liouville measure $\mathcal{L}$, then $\mathcal{L}$ is proportional to the Gibbs measure $m_{F^{su}}$ associated to the geometric potential F^{su}. Furthermore, the topological pressure $P(F^{su})$ is equal to zero.*

In particular, we also have

Corollary 4.7. *Let X be a complete Riemannian manifold with dimension at least 2 and pinched negative sectional curvature at most -1. Assume that the derivatives of the sectional curvature are uniformly bounded. If X has finite volume, then*

$$\frac{m_{F^{su}}}{m_{F^{su}}(T^1X)} = \frac{\mathcal{L}}{\mathcal{L}(T^1X)}.$$

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